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Abstract

There are various ways to regularize ill-posed operator equations in Hilbert space. If the underlying operator is accretive then Lavrentiev regularization (singular perturbation) is an immediate choice. The corresponding convergence rates for the regularization error depend on the given smoothness assumptions, and for general accretive operators these may be both with respect to the operator or its adjoint. Previous analysis revealed different convergence rates, and their optimality was unclear, specifically for adjoint source conditions. Based on the fundamental study by T. Kato, *Fractional powers of dissipative operators*. J. Math. Soc. Japan, 13(3):247–274, 1961, we establish power type convergence rates for this case. By measuring the optimality of such rates in terms on limit orders we exhibit optimality properties of the convergence rates, for general accretive operators under direct and adjoint source conditions, but also for the subclass of nonnegative selfadjoint operators.

1 Introduction

We shall consider in a Hilbert space setting *ill-posed linear operator equations* of the form

$$Au = f, \qquad f \in \mathcal{R}(A), \tag{1}$$

where $A : \mathcal{H} \to \mathcal{H}$ is a *bounded* linear operator with range $\mathcal{R}(A)$ in an *infinite-dimensional* and *separable complex Hilbert space* $\mathcal{H}$ with norm $\|\cdot\|$ and complex-valued inner product $\langle\cdot,\cdot\rangle$. The ill-posedness of equation (1) arises from the fact that the range $\mathcal{R}(A)$ is a non-closed subset of $\mathcal{H}$. Hence, for the stable approximate solution of the ill-posed equation (1), regularization methods are required when we observe, instead of the right hand side f, noisy data $f^\delta \in \mathcal{H}$ with

$$\| f - f^\delta \| \le \delta, \tag{2}$$

where $\delta > 0$ denotes the noise level.

In the sequel we restrict our considerations to the class of accretive operators, to be introduced in Section 2. Such operators allow for calculus similar to the one for nonnegative selfadjoint operators (cf., e.g., [2,

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Section 2.3]). In particular, one can use the Lavrentiev regularization as a specific form of singular perturbation, where for a regularization parameter $\gamma > 0$ the approximate solutions u_γ^δ satisfy the equation

$$(A + \gamma I)u_\gamma^\delta = f^\delta, \tag{3}$$

see the pioneering work [12] and for more general results on the method of Lavrentiev regularization the monograph [1]. For accretive operators A the operator $A + \gamma I$ is continuously invertible for all $\gamma > 0$, with operator norm bound γ^{-1}, and hence $u_\gamma^\delta \in \mathcal{H}$.

There is an immediate representation of the difference $u - u_\gamma^\delta$ by

$$u - u_\gamma^\delta = (A + \gamma I)^{-1} (Au + \gamma u - f^\delta) = \gamma (A + \gamma I)^{-1} u + (A + \gamma I)^{-1}(Au - f^\delta).$$

Clearly, under the noise assumption (2) the last term can be norm bounded by

$$\| (A + \gamma I)^{-1} (Au - f^\delta) \| \leq \delta \| (A + \gamma I)^{-1} \|_{\mathcal{L}(\mathcal{H})} \leq \frac{\delta}{\gamma},$$

where $\| \cdot \|_{\mathcal{L}(\mathcal{H})}$ denotes the operator norm of the Banach space $\mathcal{L}(\mathcal{H})$ of bounded linear operators mapping in $\mathcal{H}$. Thus, introducing the *bias* (regularization error for noise-free data) $\| \gamma (A + \gamma I)^{-1} u \|$, we arrive at the following bound for the overall regularization error

$$\| u - u_\gamma^\delta \| \leq \| \gamma (A + \gamma I)^{-1} u \| + \frac{\delta}{\gamma}. \tag{4}$$

So, in order to obtain an overall regularization error bound, consideration can be restricted to bounding the bias. Such approach has been undertaken in [9], where the functional dependence of the bias in terms of the parameter is called profile function. This can be done by imposing some smoothness on the unknown solution u, and a good portion of regularization theory is concerned with this topic.

Often smoothness is given in terms of source conditions, in most cases with respect to the selfadjoint companion A^*A of A, say as $u = (A^*A)^{p/2} v$, for some $p > 0$ and some source element $v \in \mathcal{H}$. However, for accretive operators one can directly use the operator A, since one defines fractional powers through the Balakrishnan representation, and we outline this construction in Section 2. But, along with the operator A, its adjoint A^* is also accretive, and hence we may use both types of source conditions $u = A^p v$ or $u = (A^*)^p v$, the first being the *direct* one, and the latter being the *adjoint source condition*.

Error bounds for Lavrentiev regularization and for non-selfadjoint accretive operators have been considered earlier, and we shall review some results in this direction in Section 3. Those results have in common that the best possible error rate is $\delta^{1/3}$, however, obtained under adjoint source conditions with $p = 1$. This contrasts to the selfadjoint case, and the case of direct source conditions, where this rate is obtained for $p = 1/2$, already. The goal of this study is to discuss optimal rates under adjoint source conditions, in particular to derive tight upper bounds, see Theorems 1 and 2 in Section 4. Moreover, we indicate in Section 5 that in general, i.e., for arbitrary accretive operators, these bounds cannot be improved, see Corollary 2. In this context, the class of fractional integration operators in $L^2(0, 1)$ serves as a counterexample for preventing higher bias rates. As a conclusion, Section 6 summarizes the essential results of the preceding sections with respect to the worst case bias over all source elements and all normalized accretive operators by introducing the concept of limit orders for the decay of the bias. We accomplish the study in Section 7 with some discussion and extensions with respect to two restricted classes of accretive operators, one of which is the class of nonnegative selfadjoint operators. An appendix collects proofs or sketches of proofs for presented lemmas, propositions and theorems.

2 Accretive operators and related source conditions

Definition 1 (accretive operator)**.** *A bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ is called accretive if we have*

$$\mathbf{Re}\, \langle Au, u\rangle \geq 0 \quad \textit{for all} \quad u \in \mathcal{H}. \tag{5}$$

Notice that from the very definition, the operator A is accretive if and only if this holds true for the adjoint A^*. Evidently, for a real Hilbert space $\mathcal{H}$ the concepts of accretive and monotone linear operators coincide.

For any bounded linear accretive operator $A : \mathcal{H} \to \mathcal{H}$ and each constant $\gamma > 0$, the operator $A + \gamma I : \mathcal{H} \to \mathcal{H}$ possesses a bounded inverse on $\mathcal{H}$, and we have

$$\|\, (A + \gamma I)^{-1} \,\|_{\mathcal{L}(\mathcal{H})} \leq \tfrac{1}{\gamma} \quad \text{for all} \quad \gamma > 0. \tag{6}$$

As outlined in the introduction, we shall focus on tight bounds for the bias $\|\, u - u_\gamma \,\|$, where u_γ solves the equation $(A + \gamma I)\, u_\gamma = f$. However, u_γ is practically not available since f is unknown.

Definition 2 (bias)**.** *Given a bounded linear accretive operator $A : \mathcal{H} \to \mathcal{H},\ u \in \mathcal{H}$ and a parameter $\gamma > 0$, the bias (regularization error for noise-free data) is given as*

$$\mathrm{b}_\gamma(u) := \|\, \gamma\, (A + \gamma I)^{-1}\, u \,\|.$$

As highlighted earlier, the decay rate of the bias for $\gamma \to 0$ depends on properties of the unknown solution element $u \in \mathcal{H}$. Such properties are often given in terms of source conditions, preferably of *power type*. For selfadjoint operators such powers can be defined through spectral calculus. However, for accretive operators this can alternatively as follows, and we refer to [7, Chapt. 3] for details.

Definition 3 (fractional power)**.** *For $0 < p < 1$, the fractional power A^p of a bounded linear accretive operator $A : \mathcal{H} \to \mathcal{H}$ is defined by the the improper Banach space-valued integral (Balakrishnan representation)*

$$A^p := \frac{\sin \pi p}{\pi} \int_0^\infty s^{p-1}\, (A + sI)^{-1}\, A\, ds. \tag{7}$$

For arbitrary values $p > 0$, the fractional power A^p of the operator A is defined by $A^p := A^{p - \lfloor p \rfloor} A^{\lfloor p \rfloor}$, where $\lfloor p \rfloor$ denotes the largest integer which does not exceed p.

Remark 1. As general references for fractional powers of operators we refer to [7, 11, 16]. Below, some more properties of fractional powers considered in those references will be tacitly used.

Thus for $0 < p < \infty$ and for the accretive operator A one can consider source conditions, both in direct form as $u \in \mathcal{R}(A^p)$, and in its adjoint form $u \in \mathcal{R}((A^*)^p)$.

Here we shall constrain to studying the bias under *adjoint source conditions*.

Definition 4 (direct and adjoint source conditions)**.** *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear accretive operator. The element u obeys a direct or an adjoint source condition, if there is some $0 < p < \infty$, and an element $v \in \mathcal{H}$ such that*

$$u = A^p v \tag{8}$$

or

$$u = (A^*)^p\, v, \tag{9}$$

respectively.

For both types, the direct and adjoint source conditions bounds for the bias have been obtained, earlier. It is well known that for the direct case one may resort on a specific interpolation inequality, written as

$$\| A^p x \| \le 2 \| Ax \|^p \| x \|^{1-p} \quad \text{for all} \quad x \in \mathcal{H}, \quad 0 < p \le 1. \tag{10}$$

This inequality (10) with the factor 2 on the right-hand side follows as a consequence of a careful examination of the proof of Theorem 1.1.18 in [16] and of the fact that

$$\| (A + \gamma I)^{-1} A \|_{\mathcal{L}(\mathcal{H})} \le 1 \quad \text{for all} \quad \gamma > 0.$$

In order to better understand the character of the range type source conditions (8) and (9) from Definition 4 we mention a technical lemma which formulates a relation between the ranges of the fractional powers of accretive linear operators A and A^*.

Lemma 1. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear accretive operator. Then we have for $0 < p < 1/2$*

$$\mathcal{R}(A^p) = \mathcal{R}((A^*)^p). \tag{11}$$

Moreover, we have for $0 < p \le 1$

$$\mathcal{R}(A^p) = \mathcal{R}((AA^*)^{p/2}) \quad \textit{and} \quad \mathcal{R}((A^*)^p) = \mathcal{R}((A^*A)^{p/2}). \tag{12}$$

The proof of the equality (11) is based on a seminal result by Kato [10]. Additional details are given in the appendix, where also the proof of the equality (12) can be found. Note that, for $0 < p \le 1$, as a consequence of the identities in (12) only the two types of range conditions occurring in Definition 4 are of interest, whereas source conditions for ranges $\mathcal{R}((AA^*)^{p/2})$ and $\mathcal{R}((A^*A)^{p/2})$ need not be considered separately.

Example 1 (fractional integration operators). We introduce here as a typical non-selfadjoint accretive operator the Riemann–Liouville fractional integration operator V (for details see also [16, 18]), sometimes called Volterra operator (cf. [7, Section 8.5]), defined as

$$[Vu](x) := \int_0^x u(y)dy, \qquad 0 \le x \le 1, \tag{13}$$

on the complex Hilbert space $\mathcal{H} = L^2(0,1)$, supplied with the standard L^2-norm $\| \cdot \|_{L^2(0,1)}$, and its fractional powers for exponents $0 < p < 1$ of the form

$$[V^p u](x) = \frac{1}{\Gamma(p)} \int_0^x (x-y)^{-(1-p)} u(y)dy, \qquad 0 \le x \le 1, \tag{14}$$

where Γ denotes Euler's gamma function. One easily obtains for $0 < p \le 1$ also the adjoint operators as

$$[(V^*)^p u](x) = \frac{1}{\Gamma(p)} \int_x^1 (y-x)^{-(1-p)} u(y)dy, \qquad 0 \le x \le 1. \tag{15}$$

Also for $0 < p \le 1$, along with V all operators V^p and their adjoints $(V^p)^* = (V^*)^p$ are accretive. Moreover, we note that for such p the ranges of V^p can be verified explicitly as subspaces of the Sobolev spaces $H^p(0,1)$ of fractional order (cf. [4, Theorem 2.1] and for special cases [18, Remark 18.1 and Theorem 18.3]). A similar explicit structure (cf. [5, Lemma 8]) can also be found for the ranges of $(V^*)^p$ taking into account that for a function $u \in L^2(0,1)$ with $u = V^p\, v \in \mathcal{R}(V^p)$ the function $\tilde{u} \in L^2(0,1)$ defined as $\tilde{u}(t) := u(1-t),\ 0 \le t \le 1$, obeys the condition $\tilde{u} = (V^*)^p\, \tilde{v} \in \mathcal{R}((V^*)^p)$ for $\tilde{v}(t) := v(1-t),\ 0 \le t \le 1$.

Now we turn to estimates for the bias of Lavrentiev-regularized solutions and highlight the following upper bound.

Proposition 1. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear accretive operator. Suppose that $u = A^p v,\ \| v \| \le E$, for some $0 < p \le 1$. Then we have the inequality*

$$\mathrm{b}_\gamma(u) \le 2E\gamma^p, \quad \gamma > 0.$$

Therefore, under noisy data (2) the a priori parameter choice $\gamma(\delta) \sim \delta^{1/(p+1)}$ gives

$$\| u - u^\delta_{\gamma(\delta)} \| = \mathcal{O}(\delta^{p/(p+1)}) \quad \text{as} \quad \delta \to 0 \qquad \textit{whenever} \quad u \in \mathcal{R}(A^p). \tag{16}$$

Proof. We first observe that

$$\begin{aligned} \| \gamma \, (A + \gamma I)^{-1} \, A^p v \| &= \gamma \| \, A^p \, (A + \gamma I)^{-1} \, v \, \| \\ &\le 2\gamma \| \, A \, (A + \gamma I)^{-1} \, v \, \|^p \| \, (A + \gamma I)^{-1} \, v \, \|^{1-p} \\ &\le 2\gamma \| \, v \, \| \gamma^{p-1} = 2 \| \, v \, \| \gamma^p. \end{aligned}$$

From the error bound (4) we then have

$$\| u - u^\delta_{\gamma(\delta)} \| \le 2 \, E \, \gamma(\delta)^p + \frac{\delta}{\gamma(\delta)}.$$

With the given a priori parameter choice this completes the proof. ☐

Remark 2. Up to the factor 2 this resembles the known bounds in the selfadjoint case. This also shows that the maximal decay rate for the bias achieves the order γ^p within the range $0 < p \le 1$. We note that the rate result (16) was also mentioned by Tautenhahn in [20]. For similar rates in the case of an a posteriori parameter choice applied to an iterated version of Lavrentiev regularization we refer to [17].

3 Known results for Lavrentiev regularization under adjoint source conditions

For adjoint source conditions, and for real Hilbert spaces, the bias and the overall regularization error have been treated several times.

1. In Liu and Nashed [13] it has been shown for a nonlinear setting that the convergence rate in the noise-free case

   $$\| \gamma \, (A + \gamma I)^{-1} \, u \| = \mathcal{O}(\sqrt{\gamma}) \quad \text{as} \quad \gamma \to 0 \qquad \text{whenever} \quad u \in \mathcal{R}(A^*)$$

   can be derived. This immediately yields the following proposition.

   Proposition 2. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear accretive operator. For the scheme (3) of Lavrentiev regularization with noisy data as in (2), and for the a priori parameter choice $\gamma(\delta) \sim \delta^{2/3}$ we have the convergence rate*

   $$\| u - u^\delta_{\gamma(\delta)} \| = \mathcal{O}(\delta^{1/3}) \quad \text{as} \quad \delta \to 0 \qquad \textit{whenever} \quad u \in \mathcal{R}(A^*). \tag{17}$$

2. Hofmann, Kaltenbacher, and Resmerita [8] studied Lavrentiev regularization under variational source conditions. These authors did not explicitly focus on the bias, instead they showed the convergence rate for the overall regularization error.

   Proposition 3. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear accretive operator. For each $0 < p \leq 1$, the convergence rate*

   $$\| u - u^\delta_{\gamma(\delta)} \| = \mathcal{O}(\delta^{p/(2p+1)}) \quad \text{as} \quad \delta \to 0 \qquad \textit{whenever} \quad u \in \mathcal{R}((A^*)^p)$$

   holds true for the a priori parameter choice $\gamma(\delta) \sim \delta^{(p+1)/(2p+1)}$.

Actually, for the specific question the arguments in both papers are similar, and we briefly sketch the following bound for the bias.

Proposition 4. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear accretive operator. If u obeys an adjoint source condition (9) for some $0 < p \leq 1$, then there is a constant $C_p > 0$ such that*

$$\mathrm{b}_\gamma(u) \leq C_p \gamma^{p/(p+1)}, \quad \gamma > 0.$$

Proof. We start with the interpolation inequality (10) and bound, for $u = (A^*)^p\, v$

$$\langle u, u - u_\gamma \rangle = \langle v, A^p(u - u_\gamma)\rangle \leq \| v \| \| A^p(u - u_\gamma) \| \leq 2 \| v \| \| A(u - u_\gamma) \|^p \| u - u_\gamma \|^{1-p}.$$

Now, from the definition of Lavrentiev regularization in (3), by testing with $u - u_\gamma$ and $A(u - u_\gamma)$ we derive the following two 'basic inequalities' (see Eqs. (9) & (11) with $\delta = 0$ in [8]):

$$\| u - u_\gamma \|^2 \leq \langle u, u - u_\gamma \rangle \quad \text{and} \quad \| A(u - u_\gamma) \| \leq \gamma \| u \|.$$

Combining the above inequalities this gives

$$\| u - u_\gamma \|^2 \leq 2 \| v \| \| A(u - u_\gamma) \|^p \| u - u_\gamma \|^{1-p} \leq 2 \| v \| \| u - u_\gamma \|^{1-p} \gamma^p \| u \|^p,$$

which finally implies

$$\| u - u_\gamma \| \leq (2 \| v \|)^{1/(p+1)} \, (\gamma \| u \|)^{p/(p+1)} = (2 \| v \| \| u \|^p)^{1/(p+1)} \, \gamma^{p/(p+1)},$$

which completes the proof with $C_p = (2 \| v \| \| u \|^p)^{1/(p+1)}$. ☐

This bound actually indicates the following deficit. The constant depends on the norm of the underlying solution, and hence, some tightness in the estimate may be lost. Below, we shall break new ground in order to obtain optimal bounds.

4 Tight upper bounds under adjoint source conditions

The following result, which represents one of the main ingredients in our paper, is from [10], but we formulate it in correspondence with our setting for bounded operators only.

Proposition 5 (Kato [10])**.** *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear accretive operator and $0 < p < \frac{1}{2}$. Then*

$$\| (A^*)^p u \| \leq e_p \, \| A^p u \| \qquad \textit{for all} \quad u \in \mathcal{H}, \tag{18}$$

where $e_p = \tan \frac{\pi(1+2p)}{4}$.

Remark 3.

1. It is shown in [14] that there exists an unbounded accretive operator which does not satisfy an estimate of the form (18) for the case $p = \frac{1}{2}$ and any finite constant $e_{1/2} > 0$, in general.
2. Extensions of Proposition 5 are possible in special cases, and we refer Proposition 6 in Section 7 for further details.

From Proposition 5 we immediately derive the following result.

Corollary 1. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear accretive operator, and let $0 < p < \frac{1}{2}$. Then we have*

$$\gamma \, \| \, (A + \gamma I)^{-1} (A^*)^p \, \|_{\mathcal{L}(\mathcal{H})} \le c_p \, \gamma^p \, , \qquad \gamma > 0 \, , \tag{19}$$

with $c_p = 2e_p$, where the constant e_p is taken from Proposition 5.

Proof. Evidently, we have $\| \, S^* \, \|_{\mathcal{L}(\mathcal{H})} = \| \, S \, \|_{\mathcal{L}(\mathcal{H})}$ for each bounded linear operator S mapping in $\mathcal{H}$ and moreover

$$(A^*)^p = (A^p)^*$$

(see [7, Prop. 7.0.1 (e)]), with the consequence that

$$\| \, (A + \gamma I)^{-1} (A^*)^p \, \|_{\mathcal{L}(\mathcal{H})} = \| \, A^p (A^* + \gamma I)^{-1} \, \|_{\mathcal{L}(\mathcal{H})}.$$

Based on that fact and (19) we can proceed with the estimation of the right-hand side of the latter identity. For each $z \in \mathcal{H}$ we have

$$\gamma \| \, A^p (A^* + \gamma I)^{-1} z \, \| \le e_p \gamma \| \, (A^*)^p (A^* + \gamma I)^{-1} z \, \| \le 2 e_p \gamma^p \| \, z \, \|.$$

The latter inequality follows from the interpolation inequality (10) and from estimate (6) (with $A := A^*$, respectively). □

We are now in a position to present the first main result of this paper for Lavrentiev regularization.

Theorem 1. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear accretive operator. If the element u obeys an adjoint source condition (9) with $0 < p < \frac{1}{2}$ and $\| \, v \, \| \le E$ then the bias is bounded by*

$$\mathrm{b}_\gamma(u) \le c_p E \gamma^p \, , \qquad \gamma > 0 \, .$$

Therefore, under noisy data (2) the a priori parameter choice $\gamma(\delta) \sim \delta^{1/(p+1)}$ gives

$$\| \, u - u^\delta_{\gamma(\delta)} \, \| = \mathcal{O}(\delta^{p/(p+1)}) \quad \text{as} \quad \delta \to 0 \qquad \textit{whenever} \quad u \in \mathcal{R}((A^*)^p). \tag{20}$$

Proof. We start again from the error bound (4), and we use Corollary 1. Then we have for $u = (A^*)^p \, v$, $v \in \mathcal{H}$, that

$$\| \, u - u^\delta_{\gamma(\delta)} \, \| \le c_p \, \gamma(\delta)^p \, \| \, v \, \| + \frac{\delta}{\gamma(\delta)},$$

where the constant c_p is chosen as in Corollary 1. With the given a priori parameter choice this completes the proof of the theorem. □

Remark 4. Notice that we obtain the same rates as for the direct source conditions, cf. Proposition 1 and Remark 2. However, for the adjoint source condition this applies to the limited range $0 < p < 1/2$.

We now consider the case $p = 1/2$.

Theorem 2. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear accretive operator. If the element u obeys an adjoint source condition (9) with $p = \frac{1}{2}$ and $\| v \| \le E$ then the bias is bounded by*

$$\mathrm{b}_\gamma(u) \le c\, E \, |\ln \gamma| \, \gamma^{1/2}, \qquad 0 < \gamma < \exp(-2),$$

where $c > 0$ denotes some constant. Therefore, under noisy data (2) the a priori parameter choice $\gamma(\delta) \sim (\delta / |\ln \delta|)^{2/3}$ gives

$$\| u - u^\delta_{\gamma(\delta)} \| = \mathcal{O}\left(\left(\delta \, |\ln \delta|^2 \right)^{1/3} \right) \quad \text{as} \quad \delta \to 0 \qquad \textit{whenever} \quad u \in \mathcal{R}((A^*)^{1/2}).$$

The proof is given in the appendix.

Remark 5. The authors in [1] and [8] prove convergence results $u - u^\delta_\gamma \to 0$ as $\delta \to 0$, both for a priori and a posteriori parameter choices, under the additional assumption that the solution u of (1) is a minimum-norm solution. For elements u which satisfy a source condition, either direct or adjoint, this automatically holds true, because u is in the orthogonal complement of the nullspace $\mathcal{N}(A)$. The latter is a consequence of the fact that the orthogonal decompositions $\mathcal{N}(A) \oplus \overline{\mathcal{R}(A^*)} = \mathcal{H}$ and $\mathcal{N}(A) \oplus \overline{\mathcal{R}(A)} = \mathcal{H}$ are valid for accretive operators, and taking into account that for each $p > 0$ we have $\mathcal{R}(A^p) \subset \overline{\mathcal{R}(A)}$ and $\mathcal{R}((A^*)^p) \subset \overline{\mathcal{R}(A^*)}$, cf. [7, Corollary 3.1.11].

5 Lower bounds for fractional integration operators

The fractional integration operators from Example 1 can be used here to obtain lower bounds for the decay rate of the bias functions. To this end we shall find lower bounds for specifically chosen elements, see (22), below. Precisely, we consider for real parameters q the function

$$f_q(x) := (1 - x)^q, \quad 0 \le x < 1. \tag{21}$$

For each $-\frac{1}{2} < q < \infty$ we obviously have $f_q \in \mathcal{H} = L^2(0,1)$. It follows from elementary calculus that

$$(V^*)^p f_{q-p} = \frac{\Gamma(q - p + 1)}{\Gamma(q + 1)} f_q, \quad \text{for} \quad 0 < p < \infty, \;\; q > p - \tfrac{1}{2}.$$

Thus we have, in particular, $f_q \in \mathcal{R}((V^*)^p)$ for $q > p - \frac{1}{2}$. Moreover, we consider in the sequel with $\gamma > 0$ the function

$$u_{q,\gamma} := \gamma (V + \gamma I)^{-1} f_q, \quad \text{for} \quad -\tfrac{1}{2} < q < \infty. \tag{22}$$

The function $u_{q,\gamma}$ is given as the solution to the initial value problem

$$u'_{q,\gamma}(x) + \tfrac{1}{\gamma} u_{q,\gamma}(x) = f'_q(x) \quad \text{for} \quad 0 \le x < 1, \qquad u_{q,\gamma}(0) = 1,$$

and hence we have that

$$u_{q,\gamma} = u_{\gamma,\text{hom}} + u_{\gamma,\text{par}}, \quad \text{with} \quad u_{\gamma,\text{hom}}(x) = e^{-x/\gamma}, \; u_{\gamma,\text{par}}(x) = \int_0^x e^{-(x-y)/\gamma} f'_q(y) dy.$$

We shall use this construction in several cases, for $-\frac{1}{2} < q < 0$, and for $q = n = 1, 2, \ldots$. We start with the first case.

Lemma 2. *The function $u_{q,\gamma}$ from (22), with $-\frac{1}{2} < q < 0$, satisfies the inequality $\| u_{q,\gamma} \|_{L^2(0,1)} \geq c_q \gamma^{q+1/2}$ for sufficiently small parameter $\gamma > 0$, where $c_q > 0$ denotes some constant that depends on q.*

Proof. We shall show the following bounds

$$\| u_{\gamma,\text{hom}} \|_{L^2(0,1)} \leq \sqrt{\tfrac{\gamma}{2}} \text{ for } \quad \gamma > 0, \quad \| u_{\gamma,\text{par}} \|_{L^2(0,1)} \geq c_q' \gamma^{q+1/2} \text{ for } \quad 0 < \gamma \leq \tfrac{1}{2}, \tag{23}$$

where $c_q' > 0$ denotes some constant that depends on q and which is specified below. The first estimate in (23) follows easily, and the proof of the second estimate will be given in the following. For each $\gamma \leq \frac{1}{2}$ and each $\gamma < x < 1$ we have

$$\begin{aligned} u_{\gamma,\text{par}}(x) &\geq \int_{x-\gamma}^{x} e^{-(x-y)/\gamma} f_q'(y) dy \geq f_q'(x-\gamma) \int_{x-\gamma}^{x} e^{-(x-y)/\gamma} dy \\ &= \gamma(1-e^{-1})|q|(1+\gamma-x)^{q-1}. \end{aligned}$$

From that we obtain

$$\begin{aligned} \| u_{\gamma,\text{par}} \|_{L^2(0,1)} &\geq \Big(\int_{\gamma}^{1} u_{\gamma,\text{par}}(x)^2 dx \Big)^{1/2} \geq \gamma(1-e^{-1})|q| \Big(\int_{0}^{1-\gamma} (1-x)^{2q-2} dx \Big)^{1/2} \\ &= \gamma(1-e^{-1}) \frac{|q|}{(1-2q)^{1/2}} (\gamma^{2q-1} - 1)^{1/2} \geq c_q' \gamma^{q+1/2} \end{aligned}$$

with $c_q' = (1-e^{-1}) \frac{|q|}{(2-4q)^{1/2}}$ which gives the second estimate in (23). The term $c_q' \gamma^{q+1/2}$ in (23) dominates the term $\sqrt{\gamma/2}$ as $\gamma \to 0$. This completes the proof of the lemma. □

Now we turn to the case $q = 1$. In this case we have that $f_1 \in \mathcal{R}(V^*)$.

Lemma 3. *The function $u_{1,\gamma}$ obeys $\| u_{1,\gamma} \|_{L^2(0,1)} \geq \sqrt{\gamma/16}$ for $0 < \gamma \leq \frac{1}{16}$. More generally, for each $n = 1, 2, \ldots$ there is a constant γ_0 such that $\| u_{n,\gamma} \|_{L^2(0,1)} \geq \sqrt{\gamma/16}$ provided that $0 < \gamma \leq \gamma_0$.*

Proof. With a slight abuse of notation, we again use the decomposition $u_{q,\gamma} = u_{\gamma,\text{hom}} + u_{\gamma,\text{par}}$, now with $q = 1$. Again, it is easy to see that $\| u_{\gamma,\text{hom}} \|_{L^2(0,1)} \geq \sqrt{\gamma/4}$ for, e.g., $\gamma \leq 1$. We shall show that $\| u_{\gamma,\text{par}} \|_{L^2(0,1)} \leq \gamma$.

We explicitly compute, since for $q = 1$ we have $f_1' = -1$,

$$\| u_{\gamma,\text{par}} \|_{L^2(0,1)}^2 = \int_0^1 \left(\int_0^x e^{-(x-y)/\gamma} \, dy \right)^2 \, dx = \gamma^2 \int_0^1 \left(1 - e^{-x/\gamma} \right)^2 \, dx \leq \gamma^2.$$

Since for $\gamma \leq \frac{1}{16}$ we have that $\sqrt{\gamma/4} - \gamma \geq \sqrt{\gamma/16}$, we can complete the proof. For larger n the following observation is important. The particular solutions $u_{\gamma,\text{par}}$ at level n, denoted by $u_{\gamma,\text{par}}^{(n)}$ are negative. Moreover, we find that $-n \leq f_n' \leq 0$, such that $\left| u_{\gamma,\text{par}}^{(n)} \right| \leq n \left| u_{\gamma,\text{par}}^{(1)} \right|$, which gives $\| u_{\gamma,\text{par}}^{(n)} \|_{L^2(0,1)} \leq n \| u_{\gamma,\text{par}}^{(1)} \|_{L^2(0,1)} \leq n \, \gamma$. This allows to complete the proof. □

We summarize the lower bounds

Corollary 2. *Consider the operator equation (1), now for $A := V$. The following lower bounds hold true.*

1. *For each $0 < p < \frac{1}{2}$, $\varepsilon > 0$ there exist constants $c_{p,\varepsilon} > 0$ and $\gamma_0 > 0$ such that*

$$\| \gamma (V + \gamma I)^{-1} (V^*)^p \|_{\mathcal{L}(L^2(0,1))} \geq c_{p,\varepsilon} \, \gamma^{p+\varepsilon} \quad \text{for} \quad 0 < \gamma \leq \gamma_0.$$

2. *For each integer* $n \geq 1$ *there exist constants* $c_n > 0$ *and* $\gamma_0 > 0$ *such that*

$$\| \gamma(V+\gamma I)^{-1} (V^*)^n \|_{\mathcal{L}(L^2(0,1))} \geq c_n \sqrt{\gamma} \quad \textit{for} \;\; 0 < \gamma \leq \gamma_0.$$

The impact of Corollary 2 is considered comprehensively in Section 6.

Remark 6. The conclusions made from Corollary 2 and based on the class of fractional integration operators V^p introduced in Example 1 are not limited to the specific Hilbert space $L^2(0,1)$. Taking into account that every separable infinite-dimensional complex Hilbert space $\mathcal{H}$ is isometrically isomorphic to the space ℓ^2 of infinite sequences of square-summable complex numbers, the operator $V : L^2(0,1) \to L^2(0,1)$ can be transformed in an invariant manner with respect to its properties, as $\tilde{V} = JVJ^* : \mathcal{H} \to \mathcal{H}$, to the separable Hilbert space $\mathcal{H}$ using the associated isomorphism $J : L^2(0,1) \to \mathcal{H}$. Because of isometry, also all the norm assertions carry over from V mapping in $L^2(0,1)$ to $\tilde{V}$ mapping in $\mathcal{H}$.

6 Limit orders for the bias decay of general accretive operators

We shall formalize the assertions of the previous sections as follows. Recall the notion of the bias b_γ in Definition 2 and note the function $b_\gamma(u) = b_\gamma^A(u),\ u \in \mathcal{H}$, depends on the accretive operator A. Related to this we introduce for $p > 0$ source sets

$$\mathcal{M}_p^* := \{u \in \mathcal{H} :\ u = (A^*)^p v,\ \|v\| \leq 1\}, \tag{24}$$

as well as

$$\mathcal{M}_p := \{u \in \mathcal{H} :\ u = A^p v,\ \|v\| \leq 1\}. \tag{25}$$

In this context we mention that obviously for $p > 0$

$$\sup_{\|v\|\leq 1} b_\gamma^A\left((A^*)^p v\right) = \sup_{\|v\|\leq 1} \|u_\gamma - (A^*)^p v\| = \|\gamma(A+\gamma I)^{-1}(A^*)^p\|_{\mathcal{L}(\mathcal{H})} \tag{26}$$

and

$$\sup_{\|v\|\leq 1} b_\gamma^A\left(A^p v\right) = \sup_{\|v\|\leq 1} \|u_\gamma - A^p v\| = \|\gamma(A+\gamma I)^{-1} A^p\|_{\mathcal{L}(\mathcal{H})}, \tag{27}$$

which means that (26) and (27) characterize the corresponding suprema of the bias over all normalized elements from $\mathcal{M}_p^*$ and $\mathcal{M}_p$, respectively.

To this end we consider source sets as mappings $\mathcal{V}$, which assign to the operator A the corresponding subset $\mathcal{V}(A)$ of $\mathcal{H}$. Then the *worst case bias* restricted to $\mathcal{V}$ over all normalized accretive linear operators A is given as

$$B(\gamma, \mathcal{V}) := \sup_{\substack{A \text{ accretive}, \\ \|A\|_{\mathcal{L}(\mathcal{H})} \leq 1}} \ \sup_{u \in \mathcal{V}(A)} b_\gamma^A(u), \quad \gamma > 0. \tag{28}$$

Since the set $\mathcal{V}(A)$ depends on the operator A (cf. (26) and (27)) the supremum in (28) is understood consecutively. Moreover, we mention that, for all parameters $\gamma > 0$, the estimate

$$B(\gamma, \mathcal{V}_1) \leq B(\gamma, \mathcal{V}_2) \;\text{ holds, whenever }\; \mathcal{V}_1(A) \subset \mathcal{V}_2(A) \;\text{ for all considered } A.$$

In particular, we have for $\mathcal{V} = \mathcal{V}_p^*$ with $\mathcal{V}_p^*(A) := \mathcal{M}_p^*$ and $\mathcal{V} = \mathcal{V}_p$ with $\mathcal{V}_p(A) := \mathcal{M}_p$ as a consequence of (26) and (27) that the functions

$$B_p^*(\gamma) := B(\gamma, \mathcal{V}_p^*), \qquad \gamma > 0,$$

and

$$B_p(\gamma) := B(\gamma, \mathcal{V}_p), \qquad \gamma > 0,$$

are the suprema of the operator norms $\|\gamma(A+\gamma I)^{-1}(A^*)^p\|_{\mathcal{L}(\mathcal{H})}$ and $\|\gamma(A+\gamma I)^{-1}A^p\|_{\mathcal{L}(\mathcal{H})}$, respectively, over all normalized accretive linear operators A.

Lemma 4. *Let A be accretive, $\| A\|_{\mathcal{L}(\mathcal{H})} \leq 1$. Suppose that $u \in \mathcal{M}_p^*$ and that $0 < q \leq p$. Then there is an element $w \in \mathcal{H}$, $\| w\| \leq 4$, such that $u = (A^*)^q\, w$ and we have $\mathcal{M}_p^* \subset 4\mathcal{M}_q^*$. Consequently, we have that $B_p^*(\gamma) \leq 4B_q^*(\gamma)$ for all $\gamma > 0$. In the same manner we have $B_p(\gamma) \leq 4B_q(\gamma)$ whenever $0 < q \leq p$.*

Proof. It will be convenient to temporarily introduce the operator $B := (A^*)^p$, which implies that $(A^*)^q = B^{q/p}$, and $0 < q/p \leq 1$. Clearly, we can write $u = (A^*)^p\, v = B^{q/p}v$, with $\|v\| \leq 1$. For $w := B^{1-q/p}v$ the interpolation inequality yields the estimate

$$\| w\| = \| B^{1-q/p}v\| \leq 2\| Bv\|^{1-q/p} \leq 4,$$

where the latter estimate follows from $\| B\|_{\mathcal{L}(\mathcal{H})} \leq \| (A^*)^{p-\lfloor p\rfloor}\|_{\mathcal{L}(\mathcal{H})}\| (A^*)^{\lfloor p\rfloor}\|_{\mathcal{L}(\mathcal{H})} \leq 2 \cdot 1 = 2$. This proves the first assertion.

The second is an easy consequence, since the bias scales linearly in the norm of the source element v. ◻

Next we consider the *limit order* for the decay rate of the functions $B(\gamma, \mathcal{V})$ as γ tends to zero.

Definition 5 (limit order for general accretive operators)**.** *Given a mapping $\mathcal{V}$ defined for general accretive linear operators the limit order for the worst case bias over $\mathcal{V}$ is given as*

$$\Lambda(\mathcal{V}) := \sup\left\{q \geq 0 : \quad B(\gamma, \mathcal{V}) = \mathcal{O}(\gamma^q) \quad \textit{as} \;\; \gamma \to 0\right\}. \tag{29}$$

In particular we define for the adjoint and direct source sets

$$\Lambda_p^* := \Lambda(\mathcal{V}_p^*),$$

and

$$\Lambda_p := \Lambda(\mathcal{V}_p).$$

Remark 7. Limit orders are a useful concept to characterize decay rates, and this concept proved important when studying properties of operators in several (classical) Banach spaces, we refer for the monograph [15, Chapt. 14.4], where such approach is undertaken in a much more complex context. Here we adopt the notion, although in a different context.

To understand the concept on which the limit order is based, we provide the following simple result, which is a consequence of the direct order for the functions $B(\gamma, \mathcal{V})$ from (28).

Lemma 5. *Suppose that the two mappings $\mathcal{V}_1$ and $\mathcal{V}_2$ are such that for all operators A under consideration it holds that $\mathcal{V}_1(A) \subset \mathcal{V}_2(A)$. Then*

$$\Lambda(\mathcal{V}_2) \leq \Lambda(\mathcal{V}_1).$$

Consequently, we find that

$$\Lambda_{p_1}^* \leq \Lambda_{p_2}^* \quad \textit{and} \quad \Lambda_{p_1} \leq \Lambda_{p_2} \quad \textit{whenever} \quad 0 < p_1 \leq p_2 < \infty. \tag{30}$$

Indeed, the second assertion follows easily from the first one and from Lemma 4. Thus, smaller classes yield larger limit orders, and for source sets as in (24) and (25) we see that the limit order is a non-decreasing function in p.

The main result on limit orders in the general accretive case is given next.

Theorem 3. *We have that*

$$\Lambda_p^* = \begin{cases} p, & 0 < p \le \frac{1}{2}, \\ \frac{1}{2}, & \frac{1}{2} \le p < \infty, \end{cases}$$

and

$$\Lambda_p = \begin{cases} p, & 0 < p \le 1, \\ 1, & 1 \le p < \infty. \end{cases}$$

Theorem 3 shows the limitation of the Lavrentiev regularization when taking into account adjoint source conditions with $p > 1/2$ in comparison to direct source conditions. In particular, the saturation rate $\mathcal{O}(\gamma)$ for the bias and hence $\mathcal{O}(\sqrt{\delta})$ for the overall regularization error cannot be achieved under adjoint source conditions, in general. This situation is entirely different from that of the Tikhonov regularization, where the adjoint source condition $u = A^* v$ yields the overall error rate $\mathcal{O}(\sqrt{\delta})$ (cf. [6, Corollary 3.1.3]). The proof of Theorem 3 is given in the appendix. We mention that the proof for the limit order Λ_p uses results to be formulated in Proposition 7, below.

7 Enhanced limit orders for restricted operator classes

Theorem 3 characterizes the worst case situation for the bias decay over all accretive bounded linear operators A. If, however, the set of of operators A is restricted to smaller classes, enhanced bias decay orders are possible, and below we mention two such classes.

A first class of operators occurs if the Lavrentiev regularization is applied to an operator A which is the fractional power of some accretive operator. In certain cases the limitation $p < 1/2$ for the adjoint source condition (9) can be broken, i.e., the range of admissible values of p can be extended to values $\frac{1}{2} \le p \le 1$ such that the rate (17) can be improved.

Proposition 6. *Let $A : \mathcal{H} \to \mathcal{H}$ and $T : \mathcal{H} \to \mathcal{H}$ be bounded accretive linear operators such that $A = T^\mu$ for some $0 < \mu < 1$. If we suppose that the element u obeys an adjoint source condition (9) with $0 < p < \frac{1}{2\mu}$, $p \le 1$, and for a source element $v \in \mathcal{H}$ with $\| v \| \le E$, then the bias is bounded by*

$$\mathrm{b}_\gamma(u) \le c_{\mu p} E \gamma^p,$$

where the constant $c_{\mu p}$ is taken from Corollary 1. Therefore, under noisy data (2) the a priori parameter choice $\gamma(\delta) \sim \delta^{1/(p+1)}$ gives (20).

Proof. An application of Proposition 5, with the operator A replaced by T, shows that for each $0 < p < \frac{1}{2\mu}$ we have $\| (A^*)^p u \| \le e_{\mu p} \| A^p u \|$ for each $u \in \mathcal{H}$, where the second law of exponents for fractional powers of operators is used, cf. [7, Corollary 3.1.5]. For the definition of the constant $e_{\mu p}$, see Proposition 5. We can now proceed as in the proofs of Corollary 1 and Theorem 1. □

Example 2. The Abel type fractional integration operator $A = V^\alpha : L^2(0,1) \to L^2(0,1)$ with $0 < \alpha < 1$ (see (14)) is the fractional power of the integration operator $V : L^2(0,1) \to L^2(0,1)$ introduced in Example 1, see formula (13). Proposition 6 thus applies to the present case, with $\mu = \alpha$.

As already mentioned these upper bounds yield lower bounds for corresponding limit orders, which will not be formally introduced, here. In particular, if $\mu < \frac{1}{2}$ then we can achieve the decay of the bias at a rate $\mathrm{b}_\gamma(u) \le C\gamma,\ u \in \mathcal{R}\left(A^*\right)$, but restricted to this particular class of accretive operators.

A more important second example class is obtained when confining to selfadjoint accretive linear operators, $A = A^*$, which are automatically positive semidefinite. Here, we consider the version

$$B^{sa}(\gamma, \mathcal{V}) := \sup_{\substack{A \text{ accretive}, \\ A = A^*, \|A\|_{\mathcal{L}(\mathcal{H})} \le 1}} \ \sup_{u \in \mathcal{V}(A)} b_\gamma^A(u), \quad \gamma > 0,$$

of the worst case bias (28) restricted to selfadjoint accretive operators A. We have

$$B^{sa}(\gamma, \mathcal{V}_p^*) = B^{sa}(\gamma, \mathcal{V}_p), \qquad \gamma > 0,$$

because the source sets (24) and (25) coincide.

Definition 6 (limit order for selfadjoint accretive operators)**.** *Given a mapping $\mathcal{V}$ defined for selfadjoint accretive linear operators the limit order for the worst case bias over $\mathcal{V}$ is given as*

$$\Lambda^{sa}(\mathcal{V}) := \sup \left\{ q \ge 0 : \quad B^{sa}(\gamma, \mathcal{V}) = \mathcal{O}(\gamma^q) \quad \text{as} \ \ \gamma \to 0 \right\}. \tag{31}$$

In particular we define

$$\Lambda_p^{sa} := \Lambda^{sa}(\mathcal{V}_p^*) = \Lambda^{sa}(\mathcal{V}_p).$$

Known results for Lavrentiev regularization in case of selfadjoint accretive operators A may be stated in terms of limit orders as follows.

Proposition 7. *We have that*

$$\Lambda_p^{sa} = \begin{cases} p, & 0 < p \le 1, \\ 1, & 1 \le p < \infty. \end{cases}$$

A sketch of the proof will be given in the appendix.

A Proofs

Proof of Lemma 1. If the operator A is injective then (11) follows easily from Kato [10] by considering the fractional power B^p of the unbounded operator $B := A^{-1} : \mathcal{H} \supset \mathcal{R}(A) \to \mathcal{H}$. This fractional power B^p can be defined in an indirect way as follows. Consider first the negative fractional power $B^{-p} \in \mathcal{L}(\mathcal{H})$ which may be defined by $B^{-p} := \frac{\sin \pi p}{\pi} \int_0^\infty t^{-p} \left(B + tI\right)^{-1} dt$ (see, e.g., [11, Section 14.2]). The fractional power B^p is then given by $B^p = (B^{-p})^{-1}$. It is the main result of [10] that the domains of definition of B^p and $(B^*)^p$ are identical. This finally results in (11), since we have $B^{-p} = A^p$ and a similar result for the adjoint operator.

Also, to prove (12) for injective A we note that, by symmetry, it is sufficient to verify, e.g., the second of the two equalities considered there. The basic ingredient in the proof is the Heinz–Kato inequality for unbounded maximal accretive operators in Hilbert spaces. For injective operators A we use the fact that

$$\|(A^*)^{-1} w\| = \| |A|^{-1} w \| \quad \text{for} \quad w \in \mathcal{R}(A^*), \tag{32}$$

where we use the notation $|A| := (A^*A)^{1/2}$.

For the domain of definitions of the unbounded operators $(A^*)^{-1}$ and $|A|^{-1}$ appearing in (32), we have $\mathcal{D}((A^*)^{-1}) = \mathcal{D}(|A|^{-1})$. Next we apply the Heinz–Kato inequality, cf. [19, Theorem 2.3.4], which gives $\mathcal{D}(((A^*)^{-1})^p) = \mathcal{D}((|A|^{-1})^p)$ for each $0 < p \le 1$. This proves the second identity of (12) if $\mathcal{N}(A) = \{0\}$ holds.

We now consider the case $\mathcal{N}(A) \ne \{0\}$. Since $\mathcal{N}(A)$ and the closure $M := \overline{\mathcal{R}(A)}$ of the range of A are orthogonal subspaces (see, e.g., [16, Section 1.1.2]), we may apply the previous results in both cases to the restriction $A_{|M} : M \to M$ of A to the subspace M. This finally gives (11), and also the second identity of (12), furthermore by symmetry the first one. We omit the simple but tedious details. □

Proof of Theorem 2. The proof of the theorem is similar to that of Theorem 1. We first estimate the term $\gamma\| (A+\gamma I)^{-1}(A^*)^{1/2} \|_{\mathcal{L}(\mathcal{H})}$. It follows from elementary calculus that the constant e_p from Proposition 5 satisfies $e_{1/2-\varepsilon} = \mathcal{O}(1/\varepsilon)$ as $\varepsilon \to 0$. In fact, we have $e_{1/2-\varepsilon} = \frac{\cos(\pi\varepsilon/2)}{\sin(\pi\varepsilon/2)}$ and $\frac{\sin x}{x} \ge c_1 := 1 - \pi/(8\sqrt{2})$ for $0 < x \le \frac{\pi}{4}$, so

$$e_{1/2-\varepsilon} \le \frac{1}{\sin(\pi\varepsilon/2)} \le \frac{c_2}{\varepsilon} \ \text{ for } \ 0 < \varepsilon < \tfrac{1}{2},$$

where $c_2 = 2/(c_1\pi)$. For $0 < \gamma < \exp(-2)$ we now may apply Corollary 1 with $\varepsilon = 1/|\ln\gamma| < \frac{1}{2}$ and obtain the following estimates:

$$\begin{aligned}
\gamma\| (A+\gamma I)^{-1}(A^*)^{1/2} \|_{\mathcal{L}(\mathcal{H})} &\le 2\gamma\| (A+\gamma I)^{-1}(A^*)^{1/2-\varepsilon} \|\|\, A \|_{\mathcal{L}(\mathcal{H})}^{\varepsilon} \\
&\le 4\| A \|_{\mathcal{L}(\mathcal{H})}^{\varepsilon} e_{1/2-\varepsilon}\gamma^{1/2-\varepsilon} \\
&\le 4c_2 \max\{1, \| A \|_{\mathcal{L}(\mathcal{H})}\} \tfrac{1}{\varepsilon}\gamma^{1/2-\varepsilon} \\
&= 4c_2 \max\{1, \| A \|_{\mathcal{L}(\mathcal{H})}\} \exp(1) |\ln\gamma| \gamma^{1/2}.
\end{aligned}$$

This gives the first statement of the theorem, with $c = 4c_2 \max\{1, \| A \|_{\mathcal{L}(\mathcal{H})}\} \exp(1)$. We next verify the second statement of the theorem. A combination of the error bound (4) and the first part of this theorem gives, for $u = (A^*)^{1/2} v,\ v \in \mathcal{H}$, the following,

$$\| u - u_{\gamma(\delta)}^{\delta} \| \le c\, |\ln\gamma(\delta)|\, \gamma(\delta)^{1/2} \, \| v \| + \frac{\delta}{\gamma(\delta)}.$$

With the given a priori parameter choice we now obtain the desired estimate for $\| u - u_{\gamma(\delta)}^{\delta} \|$. □

To establish the results on limit orders for general accretive or restricted classes, we first turn to proving Proposition 7. The proof of Theorem 3 will use this.

Sketch of the proof of Proposition 7. First, for $p \ge 1$ there is saturation of the bias function, which means that the decay order of the bias cannot be faster than γ, cf. [9, Thm. 4.1 (Ex, 4.3)]. This gives $\Lambda_p^{\mathrm{sa}} \le 1$ for $p \ge 1$. Next, for $0 < p < 1$ we consider the following selfadjoint operator. We choose any orthonormal basis $e_j,\ j = 1, 2, \ldots$, in $\mathcal{H}$, and consider the diagonal mapping A given by

$$u = \sum_{j=1}^{\infty} \langle u, e_j \rangle e_j \longrightarrow \sum_{j=1}^{\infty} \frac{1}{j} \langle u, e_j \rangle e_j, \quad u \in \mathcal{H}.$$

Clearly, its largest eigenvalue (norm) equals one. Now, given $0 < p < 1$ we fix some $\varepsilon > 0$, small enough, and let the constant c be given as $c := \left(\sum_{j=1}^{\infty} j^{-(1+2\varepsilon)}\right)^{-1}$. Then, the element $v := c\sum_{j=1}^{\infty} j^{-(1/2+\varepsilon)} e_j$ has norm equal to one, and it gives rise to the element $u := A^p v$. Then we refer to the study [3, Proposition 1], which asserts that the bias function $\mathrm{b}_\gamma(u)$ at element u can be lower bounded by its distribution function $F_u(\kappa\gamma)$ for some constant $\kappa > 0$. The square of the distribution function is given as

$$F_u^2(t) = \sum_{j=\lceil t^{-1}\rceil}^{\infty} |\langle u, e_j\rangle|^2 \asymp t^{2(p+\varepsilon)},$$

which gives $\mathrm{b}_\gamma(u) \geq c\gamma^{p+\varepsilon}$, for $\gamma > 0$ small enough, and hence

$$B^{sa}(\gamma, \mathcal{M}_p) \geq c\gamma^{p+\varepsilon}, \quad \text{for } \gamma > 0 \text{ small enough.}$$

This shows that $\Lambda_p^{\mathrm{sa}} \leq p + \varepsilon$. Since this holds true for $\varepsilon > 0$ small enough, we must have $\Lambda_p^{\mathrm{sa}} \leq p$. On the other hand, for selfadjoint operators the direct and adjoint cases are identical, and Proposition 1 implies $\Lambda_p^{\mathrm{sa}} = p$ in the range $0 < p < 1$. □

Proof of Theorem 3. We first consider the limit order under adjoint source conditions. Here, by monotonicity it is enough to establish that

$$\Lambda_p^* = 1/2, \text{ for } p \geq 1, \quad \text{and} \quad \Lambda_p^* = p \quad \text{for } 0 < p < 1/2. \tag{33}$$

Indeed, from $\Lambda_1^* = 1/2$ we get by monotonicity that $\Lambda_p^* \leq 1/2$ for $1/2 \leq p \leq 1$. Also, by monotonicity we find that $\Lambda_p^* \leq \Lambda_{1/2}^*$ for all $0 < p < 1/2$, and hence $\Lambda_{1/2}^* = 1/2$.

We turn to proving both assertions in (33). The lower bounds for Λ_p^* are a consequence of the upper bounds in Theorem 1. The upper bounds for the limit order are obtained from the lower bounds in Corollary 2, since by Remark 6 we can confine consideration to the fractional integration operator V. Therefore, the lower bounds in Corollary 2(1) (for $0 < p < 1/2$) yield that $\Lambda_p^* \leq p + \varepsilon$, and hence $\Lambda_p^* \leq p$. This proves the upper bounds on the right in (33). The upper bound on the left in (33) follows from Corollary 2(2), and completes the proof in the first case.

We turn to bounding the limit order under direct source conditions. Here we use that the class of selfadjoint accretive operators is a subclass of all, and we cannot distinguish between direct and adjoint source conditions on this subclass. Therefore, by monotonicity as given in Lemma 5, the upper bound from Proposition 7 is an upper bound for direct source conditions. But, the lower bounds follow from Proposition 1, which established a rate p on the whole range $0 < p \leq 1$. The proof is complete. □

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